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ARCHIMEDEAN SHINTANI FUNCTIONS ON $GL(2)$

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Contents

1.Introduction

Shintani functions for $GL(n)$ was defined by Murase and Sugano in the study of automorphic L-functions [5]. They proved the uniqueness and the existence of this function over a non-archimedean local field and obtained new kinds of integral formula for the standard L-functions as an application. Our aim in this note is the case study of archimedean Shintani functions on $GL(2)$, which is not studied in [5]. In §3, we define archimedean Shintani functions on $GL(2)$ generalizing that of Murase and Sugano. Also, our definition of this function can be considered as a generalization of the O_ξ model studied by Waldspurger [6]. Now we consider the following problems.

(1) Decide the dimension of the space of archimedean Shintani functions.
(2) Find an explicit formula of non-zero archimedean Shintani functions.

We will give an answer to these problems in §5.

2.Preliminaries

2.1. Groups and algebras. Throughout this note, E means either the field of real numbers $\mathbf{R}$ or that of complex numbers $\mathbf{C}$. Let G be the real reductive Lie group $GL(2,E)$ and θ be an involution defined by $\theta(g) = {}^t\bar{g}^{-1} (g \in G)$. We denote the set of fixed points of θ by K. Then K is a maximal compact subgroup of G and

$$K \simeq \begin{cases} O(2,\mathbf{R}) & \text{for } E = \mathbf{R}, \\ U(2) & \text{for } E = \mathbf{C}, \end{cases}$$

Moreover we define an involutive automorphism σ of G by $\sigma(g) = JgJ$ $(g \in G)$, where $J = \mathrm{diag}(-1,1)$. Then $\theta\sigma = \sigma\theta$ and the set H of fixed points of σ is equal to

$$H = \{g \in G | \sigma(g) = g\} = \left\{\mathrm{diag}(h_1,h_2) \in G \;\middle|\; h_i \in E^\times\right\} \simeq E^\times \times E^\times.$$

In particular, H is abelian subgroup of G.

Let $\mathfrak{g} = \mathfrak{gl}(2, E)$ be the Lie algebra of G. If we denote the differentials of θ and σ, again by θ and σ, then we have $\theta(X) = -{}^t\bar{X}$ and $\sigma(X) = JXJ(X \in \mathfrak{g})$. Let us write the eigenspaces of θ and σ by

$$\mathfrak{k} = \{X \in \mathfrak{g} | \theta(X) = X\}, \qquad \mathfrak{p} = \{X \in \mathfrak{g} | \theta(X) = -X\},$$
$$\mathfrak{h} = \{X \in \mathfrak{g} | \sigma(X) = X\}, \qquad \mathfrak{q} = \{X \in \mathfrak{g} | \sigma(X) = -X\}.$$

Therefore we have the decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$. Remark that $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{h}$ is that of H. Let

$$A = \left\{ a_r = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} \in G \,\middle|\, r \in \mathbf{R} \right\}, \quad \mathfrak{a} = \mathrm{Lie}(A).$$

Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$.

For a Lie algebra $\mathfrak{b}$, we denote by $\mathfrak{b}^{\mathbb{C}}$ the complexification $\mathfrak{b} \otimes_{\mathbf{R}} \mathbf{C}$ of $\mathfrak{b}$.

2.2. Representations. In this subsection, we recall parametrizations of the irreducible unitary representations of K, H and G.

Let us denote by $\hat{K}$ the set of the equivalence classes of irreducible finite dimensional representations of K. Since K is compact, the highest weight theory (*cf.* Knapp [4; Theorem 4.28]) gives a parametrization of $\hat{K}$ by the set

$$\Lambda = \begin{cases} \{(0, \varepsilon) \,|\, \varepsilon = 0, 1\} \cup \mathbf{N}, & \text{for } E = \mathbf{R}, \\ \{\lambda = (\lambda_1, \lambda_2) \,|\, \lambda_i \in \mathbf{Z}, \lambda_1 \geq \lambda_2\}, & \text{for } E = \mathbf{C}. \end{cases}$$

Let $(\tau_\lambda, V_\lambda) \in \hat{K}$ be the corresponding representation to $\lambda \in \Lambda$. Then we have

$$\dim V_\lambda = \begin{cases} \left.\begin{matrix} 1, & \text{if } \lambda = (0, \varepsilon) \\ 2, & \text{if } \lambda \in \mathbf{N} \end{matrix}\right\} & \text{for } E = \mathbf{R} \\ \lambda_1 - \lambda_2 + 1, & \text{for } E = \mathbf{C} \end{cases}$$

Next let us parametrize the totality $\hat{H}$ of the equivalence classes of irreducible unitary representations of H. To do this, we put

$$\mathcal{N}_E = \begin{cases} \{0, 1\}^2, & \text{for } E = \mathbf{R}, \\ \mathbf{Z}^2, & \text{for } E = \mathbf{C}. \end{cases}$$

For every $s = (s_1, s_2) \in \mathbf{C}^2$ and $k = (k_1, k_2) \in \mathcal{N}_E$, we define a representation η_s^k of H by

$$\eta_s^k\,(\mathrm{diag}(h_1, h_2)) = h_1^{k_1} h_2^{k_2} |h_1|^{s_1 - k_1} |h_2|^{s_2 - k_2}, \quad \mathrm{diag}(h_1, h_2) \in H.$$

Clearly $\hat{H} = \{\eta_s^k | s = (s_1, s_2) \in (\sqrt{-1}\mathbf{R})^2,\ k = (k_1, k_2) \in \mathcal{N}_E\}$.

Let $P = N_P A_P M_P$ be the Langlands decomposition of the upper triangular subgroup P of G. For every $z = (z_1, z_2) \in \mathbf{C}^2$ and $l = (l_1, l_2) \in \mathcal{N}_E$, we define σ_l on M_P and ν_z on $\mathfrak{a}_P = \mathrm{Lie}(A_P)$ by

$$\sigma_l\,(\mathrm{diag}(\varepsilon_1, \varepsilon_2)) = \varepsilon_1^{l_1} \varepsilon_2^{l_2}, \qquad \mathrm{diag}(\varepsilon_1, \varepsilon_2) \in M_P, \quad \varepsilon_i \in E^{(1)}$$
$$\nu_z\,(\mathrm{diag}(t_1, t_2)) = z_1 t_1 + z_2 t_2, \qquad \mathrm{diag}(t_1, t_2) \in \mathfrak{a}_P, \quad t_i \in \mathbf{R}.$$

Then we can construct a representation $\pi_z^l = \mathrm{Ind}_P^G(1_{N_P} \otimes \exp \nu_z \otimes \sigma_l)$ of G which we call the non-unitary principal series representation. A dense subspace of the representation space is

$$\{f \in \mathrm{C}^\infty(G) \mid f(namx) = e^{(\nu_z+\rho_E)\log a}\sigma_l(m)f(x)\}$$

with norm

$$\|f\|^2 = \int_K |f(k)|^2 dk,$$

and G acts by $\pi_z^l(g)f(x) = f(xg)$. Here ρ_E is the half sum of the roots of $(\mathfrak{a}_P, \mathfrak{g})$ positive for N_P.

If $z_i \in \sqrt{-1}\mathbf{R}$, then the representation π_z^l is irreducible and unitary. This representation is usually called the unitary principal series representation P_z^l of G. Now we put $\rho_{E,0} = \rho_E(\mathrm{diag}(1,-1))$, i.e. $\rho_{\mathbf{R},0} = 1$ and $\rho_{\mathbf{C},0} = 2$. If the parameter (z,l) satisfies $z_1 + z_2 \in \sqrt{-1}\mathbf{R}$, $-\rho_{E,0} < z_1 - z_2 < 0$ and $l_1 = l_2$, π_z^l is irreducible and infinitesimally unitary. The unitary version of this representation is called the complementary series representation C_z^l of G. The representations π_z^l belonging to these two series have the following K-types from the Frobenious reciprocity theorem;

$$\pi_z^l|_K = \begin{cases} \tau_0^{l_1} \oplus \sum_{n\in\mathbf{N}} \tau_{2n}, & \text{if } l_1 + l_2 \equiv 0 \pmod 2, \\ \sum_{n\in\mathbf{N}} \tau_{2n-1}, & \text{if } l_1 + l_2 \equiv 1 \pmod 2, \end{cases} \qquad \text{for } E = \mathbf{R},$$

$$\pi_z^l|_K = \begin{cases} \sum_{j=0}^{\infty} \tau_{(l_1+j,\ l_2-j)}, & \text{if } l_1 \geq l_2, \\ \sum_{j=0}^{\infty} \tau_{(l_2+j,\ l_1-j)}, & \text{if } l_1 < l_2, \end{cases} \qquad \text{for } E = \mathbf{C}.$$

In the case of $E = \mathbf{R}$, π_z^l contains the discrete series representation $D_{j,z_1+z_2}^l$ as a subrepresentation if the parameters satisfy $z_1 + z_2 \in \sqrt{-1}\mathbf{R}$, $z_1 - z_2 = -j - 1$ for $j \in \mathbf{Z}_{\geq 0}$, and $l_1 + l_2 \equiv j \pmod 2$. The K-types of $D_{j,z_1+z_2}^l$ are given by

$$D_{j,z_1+z_2}^l|_K = \sum_{n\in\mathbf{N}} \tau_{j+2n}.$$

Let us denote by $\hat{G}_\infty$ the set of the equivalence classes of irreducible unitary representations of G belonging to the above series. Then the unitary dual $\hat{G}$ of G consists of $\hat{G}_\infty$ together with the unitary characters (*cf.* Wallach [7]).

In the folloing sections, we use the same letter for a given representation and its underlying $(\mathfrak{g}^{\mathbf{C}}, K)$-module. Also we use the notations $s' = s_1 - s_2$ and $z' = z_1 - z_2$, for brevity.

3.Shintani functions

3.1. Shintani function. Let $\eta \in \hat{H}$. Consider C^∞-induced module $\mathrm{C}^\infty\mathrm{Ind}_H^G(\eta)$ with the representation space

$$\mathrm{C}_\eta^\infty(H\backslash G) = \{F \in C^\infty(G) \mid F(hg) = \eta(h)F(g), (h,g) \in H \times G\}$$

on which G acts by the right translation. Then $C^\infty_\eta(H\backslash G)$ has structure of a smooth G-module and of a $(\mathfrak{g}^{\mathbf{C}}, K)$-module.

On the other hand, let us take an irreducible Harish-Chandra module $\Pi^* \in \hat{G}_\infty$, and consider the intertwining space

$$\mathcal{I}_{\eta,\Pi} = \mathrm{Hom}_{(\mathfrak{g}^{\mathbf{C}},K)}(\Pi^*, C^\infty\mathrm{Ind}_H^G(\eta))$$

and its image

$$\mathcal{S}_{\eta,\Pi} = \bigcup_{T\in\mathcal{I}_{\eta,\Pi}} \mathrm{Image}(T).$$

Here $*$ means the contragredient $(\mathfrak{g}^{\mathbf{C}}, K)$-module. We call $\varphi \in \mathcal{S}_{\eta,\Pi}$ *a Shintani function of type* (η, Π).

For any finite dimensional K-module (τ, V_τ), we define $C^\infty_{\eta,\tau}(H\backslash G/K)$ by the space of smooth functions $F: G \to V_\tau$ with the property

$$F(hgk) = \eta(h)\tau(k)^{-1}F(g), \qquad (h, g, k) \in H \times G \times K.$$

Now let us take a finite dimensional K-module (τ, V_τ) and a K-equivariant map $i: \tau^* \to \Pi^*|_K$. Here τ^* is the contragredient representation of τ. Moreover let i^* be the pullback via i. Then the map

$$\mathcal{I}_{\eta,\Pi} \xrightarrow{i^*} \mathrm{Hom}_K(\tau^*, C^\infty_\eta(H\backslash G)) \cong C^\infty_{\eta,\tau}(H\backslash G/K)$$

gives the restriction of $T \in \mathcal{I}_{\eta,\Pi}$ to τ^* which we denote by $T_i \in C^\infty_{\eta,\tau}(H\backslash G/K)$. Set

$$\mathcal{S}_{\eta,\Pi}(\tau) = \bigcup_i T_i, \quad T \in \mathcal{I}_{\eta,\Pi},$$

and we call $\varphi \in \mathcal{S}_{\eta,\Pi}(\tau)$ *a Shintani function of type* $(\eta, \Pi; \tau)$.

3.2. Radial part. Let us write the centralizer and the normalizer of $\mathfrak{a}$ in $K\cap H$ by $Z_{K\cap H}(\mathfrak{a})$ and $N_{K\cap H}(\mathfrak{a})$, respectively. If we put $w_0 = \mathrm{diag}(1, -1)$, then the quotient group $W = N_{K\cap H}(\mathfrak{a})/Z_{K\cap H}(\mathfrak{a})$ has the unique nontrivial element $w_0 Z_{K\cap H}(\mathfrak{a})$.

For each pair of $\eta \in \hat{H}$ and a finite dimensional K-module (τ, V_τ), let us denote by $C^\infty_W(A; \eta, \tau)$ the space of smooth functions $\varphi: A \to V_\tau$ satisfying the following conditions;

$$\begin{cases} (1) \quad (\eta(m)\tau(m))\varphi(a) = \varphi(a), & m \in Z_{K\cap H}(\mathfrak{a}),\ a \in A, \\ (2) \quad (\eta(w_0)\tau(w_0))\varphi(a) = \varphi(a^{-1}), & a \in A, \\ (3) \quad (\eta(l)\tau(l))\varphi(1) = \varphi(1), & l \in K \cap H. \end{cases}$$

Lemma 3.1. (Flensted-Jensen [1; Theorem 4.1])

(1) $G = HAK = HA^+K$, *where* $A^+ = \{a_r \in A \,|\, r > 0\}$.

(2) *The set* $C^\infty_{\eta,\tau}(H\backslash G/K)$ *is in bijective correspondence, via restriction* A, *with the set* $C^\infty_W(A; \eta, \tau)$.

Let (τ, V_τ) and $(\tau', V_{\tau'})$ be two finite dimensional K-modules. For each $\mathbf{C}$-linear map $u: C^\infty_{\eta,\tau}(H\backslash G/K) \to C^\infty_{\eta,\tau'}(H\backslash G/K)$, we have a unique $\mathbf{C}$-linear map $\mathcal{R}(u): C^\infty_W(A; \eta, \tau) \to C^\infty_W(A; \eta, \tau')$ with the property $(uf)|_A = \mathcal{R}(u)(f|_A)$ for $f \in C^\infty_{\eta,\tau}(H\backslash G/K)$. We call $\mathcal{R}(u)$ *the radial part of* u.

4. Characterization

4.1. Shift operator. The vector space $\mathfrak{p}^{\mathbb{C}}$ becomes a K-module via the adjoint representation. Let $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}_S \oplus \mathfrak{p}_Z$ be the irreducible decomposition of $\mathfrak{p}^{\mathbb{C}}$ as a K-module, where $\mathfrak{p}_Z = (\mathfrak{p} \cap Z_{\mathfrak{g}})^{\mathbb{C}}$, $Z_{\mathfrak{g}}$ is the center of $\mathfrak{g}$, and $\mathfrak{p}_S \simeq V_\beta$ with $\beta = 2$ for $E = \mathbf{R}$ or $\beta = (1, -1)$ for $E = \mathbf{C}$.

Take an orthonormal basis $\{X_i\}$ of $\mathfrak{p}_S$ with respect to the Killing form. For a given $\eta_s^k \in \hat{H}$ and $(\tau_\lambda, V_\lambda)\hat{K}$, we define a first order gradient type differential operator $\nabla^S_{\eta_s^k, \tau_\lambda} : \mathrm{C}^\infty_{\eta_s^k, \tau_\lambda}(H\backslash G/K) \to \mathrm{C}^\infty_{\eta_s^k, \tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_S}}(H\backslash G/K)$ by

$$\nabla^S_{\eta_s^k, \tau_\lambda} f = \sum_i R_{X_i} f \otimes X_i, \qquad f \in \mathrm{C}^\infty_{\eta_s^k, \tau_\lambda}(H\backslash G/K),$$

where

$$R_X f(g) = \tfrac{d}{dt} f(g \cdot \exp(tX))\big|_{t=0}, \qquad \text{for } X \in \mathfrak{g}^{\mathbb{C}},\ \ g \in G.$$

This differential operator $\nabla^S_{\eta_s^k, \tau_\lambda}$ is called *the Schmid operator*. Now let us assume that $\lambda \in \mathbf{N}_{\geq 3}$ for $E = \mathbf{R}$ or $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ with $\lambda_1 - \lambda_2 \geq 2$ for $E = \mathbf{C}$. Then we can define *the minus shift operator*

$$\nabla^-_{\eta_s^k, \tau_\lambda} : \mathrm{C}^\infty_{\eta_s^k, \tau_\lambda}(H\backslash G/K) \to \mathrm{C}^\infty_{\eta_s^k, \tau_{\lambda-\beta}}(H\backslash G/K)$$

as the compositions of $\nabla^S_{\eta_s^k, \tau_\lambda}$ with the projector from $V_\lambda \otimes \mathfrak{p}_S$ into an irreducible component $V_{\lambda-\beta}$.

4.2. System of differential equations. Let $\Pi^* \in \hat{G}_\infty$, and let $(\tau_\lambda, V_\lambda) \in \hat{K}$ be the minimal K-type of Π. Moreover, let $\eta_s^k \in \hat{H}$ with $s = (s_1, s_2) \in (\sqrt{-1}\mathbf{R})^2$ and $k = (k_1, k_2) \in \mathcal{N}_E$. We consider a characterization of the space $\mathcal{S}_{\eta_s^k, \Pi}(\tau_\lambda)$ of Shintani functions of type $(\eta_s^k, \Pi; \tau_\lambda)$ by some differential equations.

Let $Z(\mathfrak{g}^{\mathbb{C}})$ be the center of the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$. It is well known that each element $u \in Z(\mathfrak{g}^{\mathbb{C}})$ acts on Π^*, hence on $\mathcal{S}_{\eta_s^k, \Pi}(\tau_\lambda)|_A$, as a scalar operator χ_u called an infinitesimal character (*cf.* Knapp [4; Chap.VIII §6]). Therefore we have the differential equation

$$\mathcal{R}(u)\varphi(a_r) = \chi_u \varphi(a_r) \tag{4.1}$$

for each $\varphi \in \mathcal{S}_{\eta_s^k, \Pi}(\tau_\lambda)|_A$ and $u \in Z(\mathfrak{g}^{\mathbb{C}})$.

Now let us assume that $\Pi^* = D^l_{j, z_1+z_2}$ if $E = \mathbf{R}$ or $\Pi^* = P^l_z$ such that $|l_1 - l_2| \geq 2$ if $E = \mathbf{C}$. Since τ_λ is the minimal K-type of Π, then $\tau_{\lambda-\beta}$ does not occur in the K-type of Π. Thus any element in $\mathcal{S}_{\eta_s^k, \Pi}(\tau_\lambda)$ is annihilated by the action of the minus shift operator $\nabla^-_{\eta_s^k, \tau_\lambda} : \mathrm{C}^\infty_{\eta_s^k, \tau_\lambda}(H\backslash G/K) \to \mathrm{C}^\infty_{\eta_s^k, \tau_{\lambda-\beta}}(H\backslash G/K)$, and hence, the differential equation

$$\mathcal{R}(\nabla^-_{\eta_s^k, \tau_\lambda})\varphi(a_r) = 0 \tag{4.2}$$

holds for each $\varphi \in \mathcal{S}_{\eta_s^k, \Pi}(\tau_\lambda)|_A$.

The above differential equations for $\varphi \in \mathrm{C}^\infty_W(A; \eta_s^k, \tau_\lambda)$ are necessary conditions for belonging to the space $\mathcal{S}_{\eta_s^k, \Pi}(\tau_\lambda)|_A$. But we can prove the following theorem which says that the above equations are also sufficient conditions.

Theorem 4.1. ([2; Proposition 6.1], [3; Theorem 5.3], [8; Theorem 2.4])

Let $\eta_s^k \in \hat{H}$, $\Pi^* \in \hat{G}_\infty$, and let $(\tau_\lambda, V_\lambda) \in \hat{K}$ be the minimal K-type of Π. Then the following system of differential equations characterizes the space $\mathcal{S}_{\eta_s^k,\Pi}(\tau_\lambda)|_A \subset C_W^\infty(A; \eta_s^k, \tau_\lambda)$ of Shintani functions of type $(\eta_s^k, \Pi; \tau_\lambda)$.

(1) *If $\Pi^* = P_z^l$ or C_z^l, the equations (4.1) for all $u \in Z(\mathfrak{g}^{\mathbf{C}})$.*

(2) *If $E = \mathbf{R}$ and $\Pi^* = D_{j,z_1+z_2}^l$, the equations (4.1) for $u = I$ and (4.2).*

5.Results

In view of Theorem 4.1, the space $\mathcal{S}_{\eta_s^k,\Pi}(\tau_\lambda)|_A$ of Shintani functions is the solution space of some system of differential equations in $C_W^\infty(A; \eta_s^k, \tau_\lambda)$. By the systems of equations in Theorem 4.1 and the constructions of Shintani functions via the Poisson integrals [3; §6], we can prove the following theorem.

Theorem 5.1. ([2; Theorem 6.2, 6.3], [3; Theorem 7.1])

Let $\eta_s^k \in \hat{H}$ and $\Pi^* \in \hat{G}_\infty$. Then the space $\mathcal{S}_{\eta_s^k,\Pi}$ of Shintani functions of type (η_s^k, Π) is non zero if and only if $\eta_s^k|_{Z_G} = \Pi|_{Z_G}$. Here Z_G is the center of G. Moreover, for such pair of representations (η_s^k, Π) we have

$$\dim \mathcal{S}_{\eta,\Pi} = \begin{cases} 2, & \text{if } E = \mathbf{R},\ \Pi^* = P_z^l, \text{ and } l_1 \neq l_2 \\ 1, & \text{otherwise.} \end{cases}$$

Moreover we can state an explicit formula of Shintani functions of type $(\eta_s^k, \Pi; \tau_\lambda)$ for some special cases.

Theorem 5.2. ([2; Theorem 6.2], [3; Theorem 7.2])

Let $\eta_s^k \in \hat{H}$ and $\Pi^* = P_z^l$ or $C_z^l \in \hat{G}_\infty$ with $l_1 = l_2$. Then the minimal K-type $(\tau_\lambda, V_\lambda) \in \hat{K}$ of Π is 1-dimensional. If the parameters s, z, k, and l satisfy the equations

$$s_1 + s_2 = z_1 + z_2, \quad k_1 + k_2 \equiv l_1 + l_2 \pmod 2 \ (E = \mathbf{R}), \quad k_1 + k_2 = l_1 + l_2 \ (E = \mathbf{C}),$$

then the space $\mathcal{S}_{\eta_s^k,\Pi}(\tau_\lambda)$ has a base whose radial part is given by

$$x^{\frac{\delta}{4}}(1-x)^{\frac{z'+\rho_{E,0}}{4}} {}_2F_1\left(\frac{z'+s'+\rho_{E,0}+\delta}{4}, \frac{z'-s'+\rho_{E,0}+\delta}{4}; \frac{\rho_{E,0}+\delta}{2}; x\right) v_0^\lambda,$$

with $\delta = 2|k_1 - l_1|$, $v_0^\lambda \in V_\lambda$, and the variable $x = \tanh^2 2r$. Here ${}_2F_1(a, b; c; x)$ is the Gauss's hypergeometric function.

Theorem 5.3. ([2; Theorem 6.3])

Let $\eta_s^k \in \hat{H}$ and $\Pi^* = D_{j,z_1+z_2}^l \in \hat{G}_\infty$. Then $(\tau_\lambda, V_\lambda) \in \hat{K}$ with $\lambda = j + 2$ is the minimal K-type of Π. If the parameters satisfy the equations

$$s_1 + s_2 = z_1 + z_2, \quad k_1 + k_2 \equiv l_1 + l_2 \equiv j \pmod 2,$$

the space $\mathcal{S}_{\eta_s^k,\Pi}(\tau_\lambda)$ has a base whose radial part is given by

$$u_{j+2}^s(y) v_{j+2} + (-1)^{k_2} u_{-j-2}^s(y) v_{-j-2}.$$

Here $\{v_{-j-2}, v_{j+2}\}$ is the standard basis of V_{j+2}, and

$$u_{\pm(j+2)}^s(r) = (-y)^{\frac{\pm s'-j-2}{4}}(1-y)^{\frac{j+2}{2}}$$

with the variable $y = (\frac{e^{2r}-\sqrt{-1}}{e^{2r}+\sqrt{-1}})^2$.

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